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ON THE SOLUTION OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS BY MEANS OF NUMERICAL INTEGRATION OPERATORS

by

A. V. Nesterchuk [Nesterčuk]

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SUMMARY

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13. ABSTRACT

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ON THE SOLUTION OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS BY MEANS OF NUMERICAL INTEGRATION OPERATORS*)

by

A. V. Nesterchuk [Nesterčuk]

Considered in [1,2] are several aspects of the solution of differential equations by means of numerical integration operators. Considered in the present paper are new numerical integration operators and their application to the computer solution of ordinary linear differential equations. Also examined is the problem of estimating the total error.

Given a function f(x) continuous on the segment [0,1]. We divide the segment [0,1] into n equal parts, $h = \frac{1}{n}$. We introduce the notation

$$x_0 = 0,\ x_i = ih,\ f_i = f(x_i),\ [f_i] = [f_0, f_1, \ldots, f_n], \qquad i = 0, 1, 2, \ldots, n.$$

We shall calculate $\int_0^x f(t)\,dt,\ 0 \leqslant x \leqslant 1$ from one of the quadrature formulas, e.g., by the trapezoidal formula. Then

$$\int_0^h f(t)\,dt \approx h\cdot\frac{1}{2}(f_0 + f_1) = h[f_0, f_1]\cdot\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix},$$

$$\int_0^{2h} f(t)\,dt \approx h\cdot\frac{1}{2}(f_0 + 2f_1 + f_2) = h[f_0, f_1, f_2]\cdot\frac{1}{2}\begin{bmatrix}1\\2\\1\end{bmatrix},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

*) Translated from Ukr. Mat. Zh. [Ukr. Math. J.], Vol. 17, No. 4, pp. 112-119 (1965).

$$\int_0^{nh} f(t)\,dt \approx h\cdot\frac{1}{2}(f_0+2f_1+\ldots+2f_{n-1}+f_n) = h[f_0, f_1, \ldots, f_n]\cdot\frac{1}{2}\begin{bmatrix}1\\2\\ \cdot\\ \cdot\\ \cdot\\2\\1\end{bmatrix},$$

or, in the general case,

$$\left[\int_0^{ih} f(t)\,dt\right] \approx [f_k]\cdot h\cdot A_1, \quad k=0,1,2,\ldots,i; \qquad i=0,1,2,\ldots,n, \tag{1}$$

where

$$A_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & 1 & \ldots & 1\\ 0 & 1 & 2 & 2 & \ldots & 2\\ 0 & 0 & 1 & 2 & \ldots & 2\\ 0 & 0 & 0 & 1 & \ldots & 2\\ \cdot & \cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot\\ 0 & 0 & 0 & 0 & \ldots & 1\end{bmatrix}.$$

Thus, the matrix A_1 is a difference operator on the right in the trapezoidal formula. In what follows we shall call it the numerical integration operator. Denoting $y_i = \int_0^{ih} f(t)\,dt$, equality (1) can be rewritten in the form

$$[y_i] \approx h[f_k]A_1, \quad k=0,1,2,\ldots,i; \qquad i=0,1,2,\ldots,n.$$

Here the error can be found on the basis of the equalities

$$\begin{aligned} y_1 &= \frac{1}{2}h(f_0+f_1) - \frac{1}{12}h^3 f''(\xi_1), \quad 0<\xi_1<h,\\ y_2 &= \frac{1}{2}h(f_0+2f_1+f_2) - \frac{1}{12}h^3 f''(\xi_2), \quad 0<\xi_2<2h,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ y_n &= \frac{1}{2}h(f_0+2f_1+\ldots+2f_{n-1}+f_n) - \frac{1}{12}h^3 f''(\xi_n), \quad 0<\xi_n<nh. \end{aligned} \tag{2}$$

From (2) it follows that

$$|[y_i] - h[f_k]A_1| < R_1, \tag{3}$$

where

$$R_1 = \frac{Mh^3}{12}[0, 1, 1, \ldots, 1], \qquad M = \max_{0<\xi<1} |f''(\xi)|.$$

The one-row matrix R_1 will be called the error vector. Expression (3) characterizes the error of the method. The total error is the sum of the method

error and the computational error. Apropos the total error, the following theorem holds.

Theorem 1. *If the function $f(x)$, defined and continuous on the segment $[0,1]$, has on this segment a second derivative for which the following condition is fulfilled:*

$$|f''(\xi)| < M, \quad 0 < \xi < 1,$$

then in calculating $\int_0^x f(t)\,dt,\ 0 < x < 1,$ *by means of the operator* A_1, *when the errors in the calculated values of the function* $f^*(ih)$ $(i = 0,1,\ldots,n)$ *do not exceed the quantity* $\frac{Mh^2}{12}$, *the vector of the total errors* R_i^* *has the form*

$$R_i^* = \frac{Mh^3}{12}[0, 2, 3, \ldots, n+1].$$

Proof. Let $f(x)$ be the exact and $f^*(x)$ the calculated values of the function. The total-error vector for the approximate vector $h[f_i^*]A_1$ is determined in the form of the difference

$$\left[\int_0^{kh} f(t)\,dt\right] - h[f_k^*]A_1, \qquad k = 0,1,2,\ldots,n. \tag{4}$$

We calculate the i-th component of vector (4). Taking (2) into account, we get

$$\frac{1}{2}h(f_0 + 2f_1 + \ldots + 2f_{i-1} + f_i) - \frac{1}{12}h^3 f''(\xi) - h[f_0^*, f_1^*, \ldots, f_i^*]\cdot\frac{1}{2}\begin{bmatrix}1\\2\\ \cdot\\ \cdot\\ \cdot\\2\\1\end{bmatrix} =$$

$$= \frac{1}{2}h(f_0 + 2f_1 + \ldots + 2f_{i-1} + f_i) - \frac{1}{12}h^3 f''(\xi) - \frac{1}{2}h(f_0^* + 2f_1^* + \ldots$$

$$\ldots + 2f_{i-1}^* + f_i^*) = \frac{1}{2}h[(f_0 - f_0^*) + (2f_1 - 2f_1^*) + \ldots$$

$$\ldots + 2(f_{i-1} - f_{i-1}^*) + (f_i - f_i^*)] - \frac{1}{12}h^3 f''(\xi),$$

$$i = 0,1,2,\ldots,n; \qquad 0 < \xi < ih. \tag{5}$$

By the condition of the theorem $|f_i - f_i^*| \le \frac{Mh^2}{12}$; then, from (5)

$$\left|\left(\left[\int_0^{ih} f(t)\,dt\right] - h\,[f_k^*]\,A_1\right)_i\right| < \frac{1}{2}h\cdot 2\cdot \frac{Mh^3}{12}i + \frac{1}{12}h^3 M =$$

$$= \frac{Mh^3}{12}(i+1), \qquad i = 1, 2, 3, \ldots, n. \tag{6}$$

Taking the obvious equality $\left|\left(\left[\int_0^{ih} f(t)\,dt\right] - h\,[f_k]\,A_1\right)_0\right| = 0$ and (6) into account, we can write

$$R_1^* = \frac{Mh^3}{12}[0, 2, 3, \ldots, n+1].$$

More accurate will be an operator constructed by means of second- and third degree parabolas. When x = h we use the trapezoidal formula; then

$$\int_0^h f(t)\,dt \approx \frac{1}{12}h(f_0 + f_1). \tag{7}$$

When x = 2h, by the Simpson formula we have

$$\int_0^{2h} f(t)\,dt \approx \frac{1}{3}h(f_0 + 4f_1 + f_2), \tag{8}$$

and when x = 3h, we have by the "3/8" Simpson formula

$$\int_0^{3h} f(t)\,dt \approx \frac{3}{8}h(f_0 + 3f_1 + 3f_2 + f_3). \tag{9}$$

But if x = 4h, we apply the "one-third" Simpson rule twice, and when x = 5h this same rule is applied from 0 to 2h and the "three-eighths" Simpson rule from 2h to 5h. As a result, by analogy with (1) we get

$$\left[\int_0^{ih} f(t)\,dt\right] \approx h\,[f_k]\,A_2, \qquad k = 0, 1, 2, \ldots, i; \qquad i = 0, 1, 2, \ldots, n, \tag{10}$$

where

$$A_2 = \frac{1}{24}\begin{bmatrix} 0 & 12 & 8 & 9 & 8 & 8 & \ldots & 8 \\ 0 & 12 & 32 & 27 & 32 & 32 & \ldots & 32 \\ 0 & 0 & 8 & 27 & 16 & 17 & \ldots & 16 \\ 0 & 0 & 0 & 9 & 32 & 27 & \ldots & 32 \\ 0 & 0 & 0 & 0 & 8 & 27 & \ldots & 16 \\ 0 & 0 & 0 & 0 & 0 & 9 & \ldots & 32 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 8(9) \end{bmatrix}.$$

The last diagonal element equals eight if n is even and 9 if n is odd. The error in this case can be found on the basis of the equalities

$$\begin{gathered}
y_1 = \frac{1}{2}h(f_0 + f_1) - \frac{1}{12}h^3 f''(\xi_1), \qquad 0 < \xi_1 < h, \\
y_2 = \frac{1}{3}h(f_0 + 4f_1 + f_2) - \frac{1}{90}h^5 f^{(IV)}(\xi_2), \qquad 0 < \xi_2 < 2h, \\
y_3 = \frac{3}{8}h(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3}{80}h^5 f^{(IV)}(\xi_3), \qquad 0 < \xi_3 < 3h, \\
y_4 = \frac{1}{3}h(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4) - \frac{1}{90}(f^{(IV)}(\xi_4') + f^{(IV)}(\xi_4'')), \\
0 < \xi_4' < 2h, \qquad 2h < \xi_4'' < 4h, \\
y_5 = \frac{1}{24}h(8f_0 + 32f_1 + 17f_2 + 27f_3 + 27f_4 + 9f_5) - \\
- \frac{h^5}{720}(3f^{(IV)}(\xi_5') + 27f^{(IV)}(\xi_5'')), \\
0 < \xi_5' < 2h, \qquad 2h < \xi_5'' < 5h, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{gathered} \tag{11}$$

Thus, it follows from (11) that

$$|[y_i] - h[f_k]A_2| < R_2, \tag{12}$$

where

$$\begin{gathered}
R_2 = \frac{Mh^5}{720}\{0, 1, 8, 27, 16, 35, 24, \ldots\}, \\
M = \max_{0 \le \xi \le 1}(|60h^{-2}f''(\xi)|, |f^{(IV)}(\xi)|), \\
k = 0, 1, 2, \ldots, i; \quad i = 0, 1, 2, \ldots, n.
\end{gathered}$$

Insofar as the total error is concerned, a theorem analogous to theorem 1

is fulfilled in the process of applying the operator A_2.

Theorem 2. If a function $f(x)$, given and continuous on the segment $[0,1]$, has on this segment derivatives of up to and including the fourth order and if the conditions $|60h^{-2}f''(\xi)| \leq M$, $|f^{(IV)}(\xi)| < M$, $0 < \xi \leq 1$, are fulfilled, in calculating $\int_0^x f(t)\,dt$ by means of the operator A_2, when the errors of the calculated values of the function $f^*(x)$ in each integration step h do not exceed the value $\frac{Mh^4}{720}$, the total-error vector R_2^* has the form

$$R_2^* = \frac{Mh^5}{720}\{0, 2, 10, 30, 20, 40, 30, 50, \ldots\}.$$

The proof of theorem 2 is analogous to the proof of theorem 1, and therefore we will not give it here.

Upon repeated integration by means of numerical operators, the matrices representing these operators multiply out. In fact, if we calculate $\int_0^x dt \int_0^t f(\tau)\,d\tau$, then, denoting $\int_0^t f(\tau)\,d\tau$ by $\Phi(t)$, we get from (1)

$$\left[\int_0^x \Phi(t)\,dt\right] \approx h[\Phi(t)]A_i = h^2[f_k]A_i^2, \qquad i = 1, 2. \tag{13}$$

It is easy to see that

$$\left[\underbrace{\int_0^x dt \int_0^t d\tau \ldots \int_0^v}_{m} f(u)\,du\right] \approx h^m[f_k]A_i^m, \qquad i = 1, 2. \tag{14}$$

It should be noted, however, that it is not always expedient to use formula (14), since the calculations can be made more successfully if the left-hand part of formula (14) is replaced by the Cauchy formula:

$$\int_0^{kh} d\eta_1 \int_0^{\eta_1} d\eta_2 \int_0^{\eta_2} d\eta_3 \ldots \int_0^{\eta_{m-1}} f(\eta_m)\,d\eta_m = \frac{1}{(m-1)!}\int_0^{kh} (kh-t)^{m-1} f(t)\,dt, \tag{15}$$

and then formula (1) is applied.

Now let us consider the solution of ordinary linear differential equations with constant coefficients by means of numerical integration operators.

Given the differential equation

$$ay' + by = f(x), \qquad 0 \leqslant x \leqslant 1, \tag{16}$$

where a and b are constants, with the initial condition

$$y(0) = y_0. \tag{17}$$

By numerical integration of Eq. (16) taking (1) and (17) into account we get

$$a\{y_k\} - ay_0\{1, 1, \ldots, 1\} + bh\{y_k\}A_i = h\{f_k\}A_i$$

or

$$\{y_k\}\{aI + bhA_i\} = h\{f_k\}A_i + ay_0\{1, 1, \ldots, 1\}, \qquad i = 1, 2, \tag{18}$$

where I is the unit matrix. The matrix $aI + bhA_i$ is invertible, since it is triangular and its diagonal elements are different from zero; and when $h \neq -\frac{a}{ba_{jj}}$ (a_{jj} are the diagonal elements of the matrix A_i) $\det\{aI + bhA_i\} \neq 0$. From (18) we get

$$\{y_k\} = \{h\{f_k\}A_i + ay_0\{1, 1, \ldots, 1\}\}B^{-1}, \tag{19}$$

where $B = aI + bhA_i$, $(i = 1,2,)$, is a matrix which we will call "resolving". Equality (19) is the computational formula for finding the solution to the problem (16)-(17).

Let us now consider the second-order differential equation

$$ay'' + by' + cy = f(x), \qquad x \in [0, 1], \tag{20}$$
$$y(0) = y_0, \qquad y'(0) = y_0',$$

where a, b, c are constants.

Integrating the first time we get

$$a(y' - y_0') + b(y - y_0) + c\int_0^x y\,dt = \int_0^x f(t)\,dt, \tag{21}$$

$$ay' + by + c\int_0^x y\,dt = \int_0^x f(t)\,dt + ay_0' + by_0.$$

Now we integrate (21) taking (13) into account and get

$$ay + b\int_0^x y\,dt + c\int_0^x dt\int_0^t y\,d\tau = \int_0^x dt\int_0^t f(\tau)\,d\tau + \int_0^x (ay_0' + by_0)\,dt + ay_0,$$

or

$$a[y_k] + bh[y_k]A_i + ch^2[y_k]A_i^2 = h^2[f_k]A_i^2 +$$

$$+ (ay_0' + by_0)[0, 1, 2, \ldots, n] + ay_0[1, 1, \ldots, 1], \qquad i = 1, 2;$$
$$k = 0, 1, 2, \ldots, n. \tag{22}$$

For the given equation we call the matrix

$$B = aI + bhA_i + ch^2A_i^2, \qquad i = 1, 2.$$

the "resolving" matrix.

In view of the fact that the matrix A_j is triangular, for its invertibility it is necessary to choose h such that

$$a + bha_{ll} + ch^2a_{ll}^2 \neq 0,$$

then det B $\neq$ 0 and formula (22) can be written in the form

$$[y_k] = \{h^2[f_k]A_i^2 + (ay_0' + by_0)[0, 1, 2, \ldots, n] + ay_0[1, 1, \ldots, 1]\}B^{-1}, \tag{23}$$
$$i = 1, 2; \qquad k = 0, 1, 2, \ldots, n.$$

Formula (23) will be the computational formula for finding the solution to problem (20). When integrating Eq. (21) it is also possible to use equality (15) instead of (13). We then get

$$[y_k]\{aI + bhA_i + ch^2[k - j]_d A_i\} =$$
$$= h^2[f_k][k - j]_d A_i + (ay_0' + by_0)[0, 1, 2, \ldots, n] + ay_0[1, 1, \ldots, 1],$$

where $i = 1,2$; $j = 0,1,2,\ldots,k$; $k = 0,1,2,\ldots,n$; $[k - j]_d$ is a diagonal matrix.

We proceed analogously in the solution of an m-order linear differential equation, applying m-fold integration using equality (14) or replacing by formula (15).

From (19) and (23) it is clear that the process of finding the solution of a differential equation reduces to arithmetic operations with numerical matrices, which makes this method suitable for use in the computer solution of differential equations of the indicated type, since standard programs are used to multiply

and invert matrices.

Here the process of inverting the matrix B has an appreciable effect on the accuracy of calculation. This must be done by the method indicated in [3], i.e., by introducing a correction to the elements of the matrix B^{-1}.

<u>Example</u>. Let us consider the simplest differential equation

$$y' + 2y = 8x^2 - 4x, \quad x \in [0, 1] \tag{24}$$

with initial condition $y(0) = 1$, having the obvious solution $y = 4x^2 - 6x + 3 - 2e^{-2x}$. For the solution we use the operator A_2 with $h = 0.2$. The computational formula for finding the solution of Eq.(24) has the form

$$[y_k] = [0.2\,[f_k]\,A_2 + [1, 1, \ldots, 1]]\,B^{-1},$$

where $f(x) = 8x^2 - 4x$, $B = I + 0.4A_2$, $k = 0,1,2,3,4,5$.

The solution matrix, obtained analytically, is written as

$$[y_k]_{anal} = [1,\ 0.6194,\ 0.3410,\ 0.2376,\ 0.3562,\ 0.8343];$$

obtained by the proposed method it is written

$$[y_k]_{A_2} = [1,\ 0.6205,\ 0.3408,\ 0.2365,\ 0.3552,\ 0.8331];$$

and by the Runge-Kutta method (with computational formula

$$\Delta y_i \approx \frac{1}{6}(k_1 + 4k_2 + k_3),\ k_1 = hf(x_i, y_i),\ k_2 = hf\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}\cdot k_1\right),\ k_3 = hf(x_i + h;\ y_i + 2k_2 - k_1))$$

$$[y_k]_{RK} = [1,\ 0.6139,\ 0.3369,\ 0.2334,\ 0.3512,\ 0.7246].$$

The relative errors at selected nodes are, respectively:

by the proposed method:

$$\delta_1 = 0.16\%,\ \delta_2 = 0.07\%,\ \delta_3 = 0.46\%,\ \delta_4 = 0.28\%,\ \delta_5 = 0.14\%;$$

by the Runge-Kutta method:

$$\eta_1 = 0.89\%,\ \eta_2 = 0.67\%,\ \eta_3 = 1.77\%,\ \eta_4 = 1.57\%,\ \eta_5 = 13.2\%.$$

Setting $\xi = z$ in (13), we get

$$\frac{M(z)-M^*(\bar{z})}{z-\bar{z}} = \sum_{j=0}^{\infty} [Q_j(z)+P_j(z)M(z)]^* [Q_j(z)+P_j(z)M(z)]. \tag{14}$$

Starting with relations (9) and (14), and then completely repeating the proof of lemma 4.1 from [1], we arrive at the following statement.

Lemma 3. In the completely indefinite case, for every finite domain G of the complex plane there exists a constant q_G such that

$$\left\| \sum_{j=0}^{\infty} Q_j^*(z) Q_j(z) \right\| < q_G. \tag{15}$$

Let us give without proof one more statement which describes the M(z) at any fixed point z (Im z $\neq$ 0) that correspond to all orthogonal operator distribution functions $\tau(\lambda)$ of the positive-definite function F(t).

Theorem 2. In the completely indefinite case, for fixed z (Im z $\neq$ 0) the set of M(z)

$$M(z) = O(z) + \frac{1}{z-\bar{z}} \Gamma^{1/2}(\bar{z}) U \Gamma^{1/2}(z) \tag{16}$$

corresponding to all possible orthogonal distribution functions $\tau(\lambda)$ of the function F(t) ranges over the "operator circle" K_z [1] with center

$$O(z) = \sum_{j=0}^{\infty} P_j^*(z) P_j(z) \left[\frac{1}{z-\bar{z}} E + \sum_{j=0}^{\infty} P_j^*(z) Q_j(z) \right]$$

and radii (left and right) $r_{\mathcal{I}} = \frac{1}{|z-\bar{z}|} \Gamma^{1/2}(\bar{z}),\ r_{r} = \frac{1}{|z-\bar{z}|} \Gamma^{1/2}(z),$ U ranges over all unitary operators in H.

Replacing z by ξ and ξ by $\bar{z}$ in (13) and multiplying we get

$$M(z) E_0(\xi, z) + E_1(\xi, z) = [M(z) D_0(\xi, z) + D_1(\xi, z)] M(\xi), \tag{17}$$

where

$$D_0(\xi, z) = -(\xi - z) \sum_{j=0}^{\infty} P_j^*(\bar{z}) P_j(\xi), \qquad D_1(\xi, z) = E - (\xi - z) \sum_{j=0}^{\infty} Q_j^*(\bar{z}) P_j(\xi),$$

$$E_0(\xi, z) = \xi + (\xi - z) \sum_{j=0}^{\infty} P_j^*(\bar{z}) Q_j(\xi), \qquad E_1(\xi, z) = (\xi - z) \sum_{j=0}^{\infty} Q_j^*(\bar{z}) Q_j(\xi). \tag{18}$$

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